Hodge theory in combinatorics

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Our goal in this talk is to exposit the recent work of Karim Adiprasito, June Huh, and Eric Katz on Hodge theory for matroids.

For further information, see [AHK], the January 2017 Notices of the AMS, or https://mattbaker.blog/
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A sequence $a_0, \ldots, a_d$ of real numbers is called unimodal if there is an index $i$ such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d.$$ 

There are numerous naturally-occurring unimodal sequences in algebra, combinatorics, and geometry.
Unimodal sequences

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Binomial coefficients are unimodal

Example

The sequence of binomial coefficients \( \binom{n}{k} \) for \( n \) fixed and \( k = 0, \ldots, n \) (the \( n^{th} \) row of Pascal’s triangle) is unimodal.
Binomial coefficients are log-concave

The sequence of binomial coefficients has a property even stronger than unimodality: it is log-concave, meaning that $a_i^2 \geq a_{i-1} a_{i+1}$ for all $i$.

Exercise: A log-concave sequence of positive real numbers is unimodal.
Binomial coefficients are log-concave

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**Exercise:** A log-concave sequence of positive real numbers is unimodal.
The Stirling numbers of the first kind, denoted $s(n, k)$, are the coefficients which appear when one writes falling factorials

$$(x)_n = x(x - 1) \cdots (x - n + 1)$$

as polynomials in $x$:

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k.$$  

The sequence $s(n, k)$ alternates in sign.

**Combinatorial interpretation:** $|s(n, k)|$ counts the number of permutations of $n$ elements having exactly $k$ disjoint cycles.
Stirling numbers of the first kind

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The sequence of (absolute values of) Stirling numbers of the first kind is unimodal and log-concave:

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Unimodality of signless Stirling numbers
The Stirling numbers of the second kind, denoted $S(n, k)$, invert the Stirling numbers of the first kind in the sense that

$$\sum_{k=0}^{n} S(n, k)(x)_k = x^n.$$ 

**Combinatorial interpretation:** $S(n, k)$ counts the number of ways to partition an $n$ element set into $k$ non-empty subsets.

For fixed $n$ (with $k$ varying from 0 to $n$), $S(n, k)$ is also log-concave and hence unimodal.
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Reed’s conjecture

Let $G$ be a connected finite graph. Define $\chi_G(t)$ to be the number of proper colorings of $G$ using $t$ colors. This is a polynomial in $t$, called the chromatic polynomial of $G$.

It satisfies the deletion-contraction relation:

$$\chi_G(t) = \chi_{G\setminus e}(t) - \chi_{G/e}(t).$$

 Ronald Reed conjectured in 1968 that for any graph $G$ the (absolute values of the) coefficients of $\chi_G(t)$ form a unimodal sequence.

William Tutte: “In compensation for its failure to settle the Four Colour Conjecture, [the chromatic polynomial] offers us the Unimodal Conjecture for our further bafflement.”
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Examples of chromatic polynomials

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1. If $G = T$ is a tree on $n$ vertices, the chromatic polynomial of $G$ is

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\chi_T(t) = t(t - 1)^{n-1} = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} t^k.
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2. If $G = K_n$ is the complete graph on $n$ vertices, then

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\chi_{K_n}(t) = t(t - 1) \cdots (t - n + 1) = \sum_{k=1}^{n} s(n, k) t^k.
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Conjecture (Welsh)

Let $k$ be a field, let $V$ be a vector space over $k$, and let $E$ be a finite subset of $V$. Let $f_i$ be the number of linearly independent subsets of $E$ of size $i$. Then $f_i$ is a log-concave sequence.
Reed’s conjecture was proved by June Huh in 2012.

Welsh’s conjecture was proved shortly thereafter by June Huh and Eric Katz, using a “trick” due to Brylawski and Lenz.

Our goal for the rest of the talk will be to describe a theorem about matroids which generalizes both Reed’s conjecture and Welsh’s conjecture.

The polynomial $\chi_G(t)/t$ is a special case of the characteristic polynomial of a matroid.
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Matroids

Matroids were introduced by Hassler Whitney as a combinatorial abstraction of the notion of linear independence of vectors. There are many different (“cryptomorphic”) ways to present the axioms for matroids.

Giancarlo Rota: Like many other great ideas of this century, matroid theory was invented by one of the foremost American pioneers, Hassler Whitney. His paper... conspicuously reveals the unique peculiarity of this field, namely, the exceptionally large variety of cryptomorphic definitions of a matroid, each one embarrassingly unrelated to every other and each one harking back to a different mathematical Weltanschaung. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.
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Whitney and Rota
**Definition**

(Independence Axioms) A **matroid** $M$ is a finite set $E$ together with a collection $I$ of subsets of $E$, called the *independent sets* of the matroid, such that:

(I1) The empty set is independent.

(I2) Every subset of an independent set is independent.

(I3) If $I$, $J$ are independent sets with $|I| < |J|$, then there exists $y \in J \setminus I$ such that $I \cup \{y\}$ is independent.

A maximal independent set is called a **basis**, and any two bases have the same cardinality, called the **rank** of $M$. 

**Overview of the proof of Rota’s Conjecture**

**Whitney numbers of the second kind**

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A maximal independent set is called a **basis**, and any two bases have the same cardinality, called the **rank** of $M$. 
Let $V$ be a vector space over a field $k$, and let $E$ be a finite subset of $V$. Define $\mathcal{I}$ to be the collection of linearly independent subsets of $E$.

Then $\mathcal{I}$ satisfies (I1)-(I3) and therefore defines a matroid.

Matroids of this form are called representable over $k$.

By a recent theorem of Peter Nelson, asymptotically 100% of all matroids are not representable over any field.
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Graphic matroids

Let $G$ be a connected finite graph, let $E$ be the set of edges of $G$, and let $\mathcal{I}$ be the collection of all subsets of $E$ which do not contain a cycle. Then $\mathcal{I}$ satisfies (I1)-(I3) and hence defines a matroid $M(G)$, which is representable over every field. By a theorem of Whitney, if $G$ is 3-connected then $M(G)$ determines the graph up to isomorphism.
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Spans

We defined matroids in terms of independent sets, which abstract the notion of linear independence. We now abstract the notion of span.

Definition

(Closure Axioms) A matroid $M$ is a finite set $E$ together with a function $\text{cl} : 2^E \to 2^E$ such that for all $X, Y \subseteq E$ and $x, y \in E$:

(S1) $X \subseteq \text{cl}(X)$.
(S2) If $Y \subseteq X$ then $\text{cl}(Y) \subseteq \text{cl}(X)$.
(S3) $\text{cl}(\text{cl}(X)) = \text{cl}(X)$.
(S4) If $q \in \text{cl}(X \cup \{p\})$ but $q \notin \text{cl}(X)$, then $p \in \text{cl}(X \cup \{q\})$.

The exchange axiom (S4) captures an intuitive feature of “geometries”. For example, if $L$ is a line in $\mathbb{P}^{r+1}_k$ and $p, q \in \mathbb{P}^{r+1}_k \setminus L$, then $q$ lies in the span of $L \cup \{p\} \iff p$ lies in the span of $L \cup \{q\} \iff p, q, L$ are coplanar.
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- **(S2)** If $Y \subseteq X$ then $\text{cl}(Y) \subseteq \text{cl}(X)$.
- **(S3)** $\text{cl}(\text{cl}(X)) = \text{cl}(X)$.
- **(S4)** If $q \in \text{cl}(X \cup \{p\})$ but $q \not\in \text{cl}(X)$, then $p \in \text{cl}(X \cup \{q\})$.

The **exchange axiom** (S4) captures an intuitive feature of “geometries”. For example, if $L$ is a line in $\mathbb{P}^{r+1}_k$ and $p, q \in \mathbb{P}^{r+1}_k \setminus L$, then $q$ lies in the span of $L \cup \{p\} \iff p$ lies in the span of $L \cup \{q\} \iff p, q, L$ are coplanar.
Spans

We defined matroids in terms of independent sets, which abstract the notion of linear independence. We now abstract the notion of *span*.

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A subset $X$ of $E$ is called a flat if $X = \text{cl}(X)$. The closure of $X$ is the intersection of all flats containing $X$.

**Example**

(Representable matroids) Let $V$ be a vector space and let $E$ be a finite subset of $V$. A subset $F$ of $E$ is a flat of the corresponding matroid if and only if there is no vector in $E \setminus F$ contained in the linear span of $F$.

The closure operator can be defined in terms of independence by declaring that $\text{cl}(X)$ equals $X$ together with all $x \in E$ such that $X \cup \{x\}$ is not independent.
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The Vámos matroid

Let $E$ be the 8 vertices of the cuboid below. Define the flats to be $\emptyset$, $E$, and the vertices, edges, and faces shown in the picture.

This determines a matroid called the Vámos matroid which is not representable over any field.
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A matroid $M$ is called \textbf{simple} if every dependent set has size at least 3. Equivalently, $M$ is simple if and only if the empty set and all singletons are closed.

A simple matroid is also called a \textit{combinatorial geometry}.

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Every matroid has a canonical simplification.
The set \( \mathcal{L}(M) \) of flats of a matroid \( M \) together with the inclusion relation forms a **lattice**, i.e., a partially ordered set in which every two elements \( x, y \) have both a **meet** \( x \wedge y \) and a **join** \( x \vee y \).

Indeed, if \( X \) and \( Y \) are flats then we can define \( X \wedge Y \) as the intersection of \( X \) and \( Y \) and \( X \vee Y \) as the closure of the union of \( X \) and \( Y \).

If \( F \) is an element of a lattice \( L \), we define the **rank** \( r(F) \) to be the maximal length \( \ell \) of a chain \( F_0 \subset F_1 \subset \cdots \subset F_\ell = F \) in \( L \).

If \( L = \mathcal{L}(M) \) for a matroid \( M \), \( r(M) \) coincides with the rank of \( M \).
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The Möbius function of a poset

There is a far-reaching extension of the Inclusion-Exclusion Principle which holds in an arbitrary poset $P$.

There is a unique function $\mu_P : P \times P \rightarrow \mathbb{Z}$, called the Möbius function of $P$, satisfying:

- $\mu_P(x, y) = 0$ if $x$ and $y$ are incomparable.
- $\mu_P(x, x) = 1$.
- If $x < y$ then $\mu_P(x, y) = -\sum_{x \leq z < y} \mu_P(x, z)$.

The Möbius Inversion Formula states that if $f : P \rightarrow \mathbb{Z}$ is any function and $g(y) = \sum_{x \leq y} f(x)$, then

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If \( L \) is a geometric lattice, the Möbius function of \( L \) is non-zero and alternates in sign:

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Theorem (Rota)

If $x \leq y$ in $L$ then

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The deletion-contraction formula for $\chi_G(t)$ suggests an extension to arbitrary matroids. This can be made to work, but it is more convenient to proceed as follows.

**Definition**

Let $M$ be a simple matroid with lattice of flats $L$. The characteristic polynomial of $M$ is

$$\chi_M(t) = \sum_{F \in L} \mu_L(\emptyset, F) t^{r(M) - r(F)}.$$
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Rota’s conjecture

The characteristic polynomial of $M$ is monic of degree $r = r(M)$, so we can write

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\chi_M(t) = t^r + w_1(M)t^{r-1} + \cdots + w_r(M).
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By Rota’s theorem, the coefficients of $\chi_M(t)$ are nonzero and alternate in sign.

The numbers $w_k(M)$ are called the Whitney numbers of the first kind for $M$.

Theorem (Rota’s Conjecture, Adiprasito–Huh–Katz 2015)

For any simple matroid $M$, the sequence $|w_k(M)|$ is log-concave and hence unimodal.
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Preliminary reductions

What Adiprasito, Huh, and Katz actually study is the so-called reduced characteristic polynomial $\tilde{\chi}_M(t) = \chi_M(t)/(t-1)$, which is a "projective analogue" of $\chi_M(t)$. Its degree is $d := r - 1$.

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The Chow ring of a matroid

Let $\mathcal{F}$ be the poset of non-empty proper flats of $M$.

The Chow ring $A^*(M)$ is the quotient of the polynomial ring $\mathbb{Z}[x_F]_{F \in \mathcal{F}}$ by the following two kinds of relations:

1. (CH1) For every $a, b \in E$,

$$\sum_{F \ni a} x_F = \sum_{F \ni b} x_F.$$

2. (CH2) $x_F x_{F'} = 0$ whenever $F$ and $F'$ are incomparable.

This can be given the structure of a graded ring in which the generators $x_F$ have degree one. There is a natural isomorphism $\deg : A^d(M) \to \mathbb{Z}$. 

Matt Baker

Hodge theory in combinatorics
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Connection to algebraic geometry

The definition of the Chow ring $A^*(M)$ is motivated by work of Feichtner and Yuzvinsky, who noted that when $M$ is realizable over $\mathbb{C}$, the ring $A^*(M)$ coincides with the usual Chow ring of a smooth projective variety $Y_M$ associated to $M$.

For experts: $Y_M$ is the “wonderful compactification” of the complement of the hyperplane arrangement associated to $M$.

In this case, one can use the so-called Hodge-Riemann relations from algebraic geometry, applied to the smooth $d$-dimensional projective algebraic variety $Y_M$, to prove the Rota conjecture for $M$. This is the basic idea in the earlier paper of Huh–Katz.
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A difficult chasm

From the introduction to [AHK]:

While the Chow ring of $M$ could be defined for arbitrary $M$, it was unclear how to formulate and prove the Hodge–Riemann relations... We are nearing a difficult chasm, as there is no reason to expect a working Hodge theory beyond the case of realizable matroids. Nevertheless, there was some evidence for the existence of such a theory for arbitrary matroids.

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Ample classes

In order to formulate precisely the main theorem of [AHK] we need a combinatorial analogue of ample divisors in algebraic geometry. The connection goes through strictly submodular functions.

A function $c : 2^E \to \mathbb{R}_{\geq 0}$ is called strictly submodular if $c(E) = c(\emptyset) = 0$ and $c(A \cup B) + c(A \cap B) < c(A) + c(B)$ whenever $A, B$ are incomparable subsets of $E$.

Each such $c$ gives rise to an element

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\ell = \ell(c) = \sum_{F \in \mathcal{F}} c(F) x_F \in A^1(M)_\mathbb{R}
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which acts by multiplication as a linear operator

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Each such \( c \) gives rise to an element

\[
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which acts by multiplication as a linear operator

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The Main Theorem of Adiprasito–Huh–Katz

Theorem (Adiprasito–Huh–Katz, 2015)

Let $M$ be a matroid of rank $r = d + 1$, let $\ell \in A^1(M)_\mathbb{R}$ be ample, and let $0 \leq k \leq \frac{d}{2}$. Then:

1. (Poincaré duality) The natural multiplication map gives a perfect pairing $A^k(M) \times A^{d-k}(M) \to A^d(M) \cong \mathbb{Z}$.

2. (Hard Lefschetz Theorem) Multiplication by $\ell^{d-2k}$ determines an isomorphism $L : A^k(M)_\mathbb{R} \to A^{d-k}(M)_\mathbb{R}$.

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Combinatorial Hodge theory implies Rota’s Conjecture

Let $\alpha(e) = \sum_{e \in F} x_F$ and $\beta(e) = \sum_{e \not\in F} x_F$. The images of $\alpha(e)$ and $\beta(e)$ in $A^1(M)$ do not depend on $e$, and are denoted by $\alpha$ and $\beta$, respectively.

The following result is proved using Weisner’s theorem on the Möbius function of a lattice:

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Let $\bar{\chi}_M(t) := \chi_M(t)/(t-1)$ be the reduced characteristic polynomial of $M$, and write $\bar{\chi}_M(t) = m_0 t^d - m_1 t^{d-1} + \cdots + (-1)^d m_d$. Then

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for all $k = 0, \ldots, d$.

Although $\alpha$ and $\beta$ are not ample, one may view them as a limit of ample classes. Together with the Hodge-Riemann relations for $k = 0, 1$ one then deduces Rota’s conjecture in a formal way.
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High-level overview of the strategy for proving the Main Theorem

The proof of the Main Theorem is inspired by Peter McMullen’s combinatorial proof of Richard Stanley’s “g-theorem”, which describes the possible sequences which can arise as the number of faces of each dimension of simple polytopes.

Stanley’s proof used the Hard Lefschetz Theorem for certain classes of toric varieties. McMullen’s proof reduces the theorem for arbitrary simple polytopes to the case of simplices using the “flip connectivity” of simple polytopes of given dimension.

Using a suitable notion of “matroidal flips” and an elaborate inductive procedure, the authors of [AHK] eventually reduce the Main Theorem to cases where the assertions can be directly verified “by hand”.

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The Kähler Package

In the work of Adiprasito–Huh–Katz, one needs only a very special case of the Hodge–Riemann relations to deduce Rota's conjecture.

So is the Main Theorem overkill if one just wants a proof of Rota's conjecture?

No: Poincaré Duality, the Hard Lefschetz Theorem, and the Hodge–Riemann relations tend to come bundled together in what June Huh has dubbed the Kähler package.

This is also the case, for example, in the de Cataldo–Migliorini approach to Hodge theory, in the work of McMullen and Fleming–Karu on Hodge theory for simple polytopes, and in the recent work of Elias–Williamson on Soergel bimodules.
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Conjecture (Welsh)

The Whitney numbers of the second kind form a log-concave, and hence unimodal, sequence for every simple matroid $M$.

Conjecture (Dowling and Wilson’s “Top-Heavy” conjecture)

Let $M$ be a simple matroid of rank $r$. Then for all $k < r/2$ we have $W_k(M) \leq W_{r-k}(M)$.

The top-heavy conjecture can be viewed as a vast generalization of the de Bruijn-Erdös theorem that every non-collinear set of points $E$ in a projective plane determines at least $|E|$ lines.
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**Theorem (Huh–Wang, 2016)**

For all matroids $M$ representable over some field $k$:

1. *The first half of the sequence of Whitney numbers of the second kind is unimodal*, i.e., $W_1(M) \leq W_2(M) \leq \cdots \leq W_{\lfloor r/2 \rfloor}(M)$.
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Unlike the situation with Whitney numbers of the first kind, the projective algebraic variety $Y_M$ which one associates to $M$ in this case is highly singular; thus instead of invoking the Kähler package for smooth projective varieties, Huh and Wang have to use analogous but much harder results about intersection cohomology.

To extend the arguments of Huh and Wang to the non-representable case, a first significant challenge would be to construct a combinatorial model for the intersection cohomology of the variety $Y_M$. 
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