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DISSECTION OF THE HYPERCUBE INTO SIMPLEXES

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ABSTRACT. A generalization of Sperner's Lemma is proved and, using extensions of p-adic valuations to the real numbers, it is shown that the unit hypercube in n dimensions can be divided into m simplexes all of equal hypervolume if and only if m is a multiple of $n!$. This extends the corresponding result for $n = 2$ of Paul Monsky.

The question as to whether a square can be divided into an odd number of (nonoverlapping) triangles all of the same area was answered in the negative by Thomas [3] (if all the vertices of the triangles are rational numbers), and in general by Monsky [2].

In this note we generalize Monsky's result to n dimensions and prove the following:

THEOREM. *Let C be the unit hypercube in n dimensions. Then C can be divided into m simplexes all of equal hypervolume if and only if m is a multiple of $n!$.*

The proof is divided into two parts. In the first we obtain a slight generalization of Sperner's Lemma while the second employs extensions of p-adic valuations to the real numbers.

Let R be an n-polytope in n-space. A simplicial decomposition of R is a division of R into simplexes such that if v is a vertex of some simplex on the boundary of the simplex S, then v is a vertex of S. We consider a simplicial decomposition of R in which each vertex of a simplex is labeled p_i for some i, $0 \leqslant i \leqslant n$, and we call the k-simplex S a complete k-simplex if the vertices of S are labeled $p_0, p_1, \ldots, p_k$.

LEMMA 1 (SPERNER'S LEMMA). *Consider a simplicial decomposition of an n-polytope R in which each vertex is labeled p_i, $0 \leqslant i \leqslant n$. Then the number of complete n-simplexes is odd if and only if the number of complete $(n-1)$-simplexes on the boundary of R is odd.*

PROOF. Note that every complete $(n-1)$-simplex on the boundary of R occurs in one n-simplex while all other complete $(n-1)$-simplexes occur in two n-simplexes. Also, a complete n-simplex has precisely one complete $n-1$ dimensional face, while an "incomplete" n-simplex has 0 or 2 such faces.

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From this, the conclusion of Sperner's Lemma follows easily.

We now consider decomposition of an n-polytope into simplexes, which may not be simplicial (i.e. a vertex of one simplex may be on a face, but not a vertex, of an adjacent simplex).

LEMMA 2. *Consider a decomposition of an n-polytope R into simplexes in which each vertex is labeled p_i, $0 \leqslant i \leqslant n$, and such that any k-dimensional affine subspace which contains vertices labeled p_i, for all $0 \leqslant i \leqslant k$ contains no vertex labeled p_i with $i > k$. Then the number of complete n-simplexes is odd if and only if the number of complete $(n - 1)$-simplexes on the boundary of R is odd.*

The proof is by induction on n. Since for $n = 0, 1$, any decomposition into simplexes is a simplicial decomposition, the result follows from Sperner's Lemma.

Assume the result for k-polytopes with $k < n$. We first show that the number of simplexes in the decomposition of R which have a complete $(n - 1)$-simplex as a face internal to R is even. Let T be an $(n - 1)$-polytope interior to R such that all the vertices of T are vertices of simplexes on both sides of the hyperplane of T. Then T can be considered as the union of two sets of $(n - 1)$-simplexes from the decomposition of R on the two sides of T. By induction, the number of complete $(n - 1)$-simplexes in these two decompositions have the same parity. This shows that the number of simplexes which have a complete $(n - 1)$-simplex as a face internal to R is even and the proof of the lemma is completed as was the proof of Sperner's Lemma.

Turning now to the p-adic valuations we use $|x|_p$ to represent the valuations at x and recall that if x is rational, $x = p^t(a/b)$ where $(a, p) = (b, p) = 1$ then $|x|_p = p^{-t}$, while $|0|_p = 0$. It is easy to show that if x and y are rational $|xy|_p = |x|_p\, |y|_p$ and $|x + y|_p \leqslant \max(|x|_p, |y|_p)$ with equality if $|x|_p \neq |y|_p$. It is known that this p-adic valuation can be extended to the reals (Theorem 1.2, [**1**, p. 121]) and we use the same notation for the extension.

Fix the prime p, let $\|x\| = |x|_p$, and separate the points $(x_1, \dots, x_n)$ in space into $n + 1$ sets $P_0, P_1, \dots, P_n$ as follows:

$$(x_1, \dots, x_n) \in P_0 \quad \text{if } \|x_i\| < 1 \text{ for all } i,$$

$$(x_1, \dots, x_n) \in P_k \quad \text{if } \|x_k\| \geqslant 1, \quad \|x_k\| > \|x_i\| \text{ for } i < k$$
$$\text{and } \|x_k\| \geqslant \|x_i\| \text{ for } k < i.$$

Suppose $(x_1, \dots, x_n) \notin P_0$ and $k \geqslant 1$. Then $(x_1, \dots, x_n) \in P_k$ if $\|x_k\| = \max\|x_j\|$, and k is the smallest index for which this equality holds. Note that if $(x_1, \dots, x_n) \in P_k$ with $k \neq 0$ then $\|x_k\| \geqslant 1$. It follows easily that each P_k is stable under translation by elements of P_0.

We next show that a k-dimensional affine subspace cannot contain points from each P_i, $0 \leqslant i \leqslant k$, and from some additional P_l. Suppose the contrary, and let the point from P_j have coordinates $(x_{1j}, \dots, x_{nj})$. (By the above we may, and do, assume the origin is the point in P_0.) However the p-adic valuation of the determinant

$$\begin{bmatrix} x_{11} & \cdots & x_{k1} & x_{l1} \\ x_{12} & \cdots & x_{k2} & x_{l2} \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ x_{1k} & \cdots & x_{kk} & x_{lk} \\ x_{1l} & \cdots & x_{kl} & x_{ll} \end{bmatrix}$$

is the same as $\|x_{11}x_{22} \ldots x_{kk}x_{ll}\|$ because it has the largest value of all terms in the expansion. Thus, the determinant is not zero and this contradicts our assumption and shows that the P_i define a labelling satisfying the hypothesis of Lemma 2. Using the same idea again, we see that if α is the highest power of the prime p which divides $n!$, then the p-adic valuation of the hypervolume of a complete n-simplex is greater than or equal to α since $|x_{11} \ldots x_{nn}|_p > 1$ and the hypervolume of the simplex is $(1/n!)\det(x_{ij})$ where x_{ij} is the ith coordinate of the point in P_j. (We assume that the origin is the point in P_0.)

Finally, since for each k there is only one k-dimensional face of the unit hypercube C which contains points from $P_0, P_1, \ldots, P_k$, it follows easily from Lemma 2 that if the unit hypercube C is divided into m simplexes, there must be an odd number of complete n-simplexes. Let S be a complete n-simplex. Then the hypervolume of S is $1/m$ and from the above $|S|_p = |1/m|_p > \alpha$ where α is the highest power of p dividing $n!$. That is, $m = p^\alpha \cdot t$ for some integer t. Since this is true for all primes p, m must be a multiple of $n!$. It is clear that one can realize any multiple of $n!$: divide C into $n!$ simplexes of the same volume, and take the decomposition obtained by dividing a one-dimensional face of each simplex into l equal parts, which yields $l \cdot n!$ simplexes of equal volume. This completes the proof of the theorem.

References

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